Published by Institute of Physics Publishing for SISSA

Received: *October 24, 2005*
Revised: *March 1, 2006*
Accepted: *March 31, 2006*
Published: *April 4, 2006*

Brane bulk couplings and condensation from REA fusion*

Jacek Pawełczyk[a] **and Rafał R. Suszek**[ab†]

[a] *Institute of Theoretical Physics, Warsaw University*
ul. Hoża 69, PL-00-681 Warsaw, Poland
[b] *King's College London*
Strand, London WC2R 2LS, U.K.
E-mail: Jacek.Pawelczyk@fuw.edu.pl, rafal.suszek@ens-lyon.fr

Abstract: The physical meaning of the Reflection Equation Algebras of [1] is elucidated in the context of Wess–Zumino–Witten D-brane geometry, as determined by couplings of closed-string modes to the D-brane. Particular emphasis is laid on the rôle of algebraic fusion of the matrix generators of the Reflection Equation Algebras. The fusion is shown to induce transitions among D-brane configurations admitting an interpretation in terms of RG-driven condensation phenomena.

Keywords: D-branes, Quantum Groups, Conformal Field Models in String Theory, Non-Commutative Geometry.

*Work supported by Polish State Committee for Scientific Research (KBN) under contract 2 P03B 001 25 (2003-2005). R.R.S. was additionally supported by the Marie Curie Fellowship at King's College London.

†Current address: Laboratoire de Physique, École Normal Supérieure de Lyon; 46, allée d'Italie, 69364 Lyon, France.

Contents

1. Introduction

Physics of D-branes has long been a subject of intense study. Of particular interest are D-branes on compact WZW manifolds. They provide interesting examples of D-brane behaviour in non-trivial backgrounds with fluxes. But even for these, highly symmetric, cases the full BCFT analysis is rather complicated [2] and stands in shocking contrast with the very simple matrix model of D-brane condensation advocated in [3]. There is a price to pay for the simplicity of the latter — it cannot describe all D-branes on the group manifold. Some years ago, a matrix model based on quantum-group symmetries was advanced [1]. It seems to work properly for all D-branes but many of its features are still mysterious. The model uses the celebrated Reflection Equation (RE) [4] and its representation theory to derive D-brane properties. On the one hand, the RE encodes a quantum-group version of the familiar pattern of bulk symmetry breakdown of the underlying WZW model resulting from the introduction of a maximally symmetric boundary — this is the particular aspect of it emphasised in [5] (the quantum group of interest is the Drinfel'd–Jimbo deformation of the universal enveloping algebra, suggested by certain fundamental structures of the associated CFT). On the other hand, it defines a quantisation of the distinguished Poisson structure on the target Lie group of the WZW model, first elucidated by Semenov-Tian-Shansky in [6], compatible with the foliation of the group manifold by conjugacy classes (to be wrapped by the maximally symmetric D-branes) and forming part of the canonical structure of the boundary WZW model itself.[1]

In this paper, we shall follow this track and analyse fusing properties of the matrices entering the RE, which — as it turns out — shed some light on the physical content of the algebra. It appears that there are two types of fusion. We shall show how both of

[1]We refer the reader to [7] for a lucid exposition and comprehensive bibliography.

JHEP04(2006)009

them lead to some known D-brane properties. As we shall see, the first type of fusion (which we dub the Bound-State Fusion — the BSF) can be interpreted as describing a process of formation of extended D-branes out of D0-branes. It also chooses a particular set of representations of the RE as the physically relevant ones. The second type (to be called the Bulk-Weight Fusion — the BWF) is just the standard representation-theoretic fusion of the function algebra on a given D-brane. Mastering this last kind of fusion is necessary to keep control of functions on a given D-brane and yields couplings of gravity to the D-brane. Our study thus takes us one step beyond the purely geometric framework developed heretofore and enables us to make contact with the rich stringy physics of the parent WZW models.

2. A lightning review of the algebraic setup

We begin by recapitulating the essential aspects of the quantum matrix models studied in [1, 8, 9]. The reader is urged to consult the original papers for details.

The central element of the quantum-group-covariant approach to the study of non-classical D-brane geometry in compact (simple-)Lie-group targets is the Reflection Equation (RE) [4]:

$$R_{21}^{\Lambda_1,\Lambda_2} M_1^{\Lambda_1} R_{12}^{\Lambda_1,\Lambda_2} M_2^{\Lambda_2} = M_2^{\Lambda_2} R_{21}^{\Lambda_1,\Lambda_2} M_1^{\Lambda_1} R_{12}^{\Lambda_1,\Lambda_2}, \tag{2.1}$$

written for operator-valued matrices[2] $M^{\Lambda_{1,2}} \in \mathrm{End}(V_{\Lambda_{1,2}}) \otimes \mathrm{REA}_q(\mathfrak{g})$, with $\mathrm{REA}_q(\mathfrak{g})$ the (abstract) Reflection Equation Algebra (REA) and $V_{\Lambda_{1,2}}$ two irreducible modules of $\mathcal{U}_q(\mathfrak{g})$, labeled by the respective highest weights Λ_1, Λ_2. Here, the deformation parameter is $q = e^{\frac{\pi i}{\kappa + g^\vee(\mathfrak{g})}}$, a value suggested by a detailed analysis of the (B)CFT of the WZW models of interest, and $R_{12}^{\Lambda_1,\Lambda_2} = (\pi_{\Lambda_1} \otimes \pi_{\Lambda_2})(\mathcal{R}_{12})$ is the suitably represented[3] $\mathcal{R}$-matrix of the Drinfel'd–Jimbo quantum group $\mathcal{U}_q(\mathfrak{g})$. The RE (2.1) is readily verified to induce a $\mathcal{U}_q(\mathfrak{g}^L \times \mathfrak{g}^R)_\mathcal{R}$-module structure on $\mathrm{REA}_q(\mathfrak{g})$ under which M^Λ transform as elements of the tensor module $V_\Lambda^{(L)} \otimes V_{\Lambda^+}^{(R)}$ (here, $\mathcal{U}_q(\mathfrak{g}^L \times \mathfrak{g}^R)_\mathcal{R}$ is just $\mathcal{U}_q(\mathfrak{g}^L) \otimes \mathcal{U}_q(\mathfrak{g}^R)$ as an algebra, with a suitably twisted coalgebra structure [1]) - indeed, the bichiral transformations:

$$(u_L \otimes u_R) \triangleright M_{ij}^\Lambda = \pi_\Lambda(S u_L)_{ik} M_{kl}^\Lambda \pi_\Lambda(u_R)_{lj}, \tag{2.2}$$

with S the antipode of $\mathcal{U}_q(\mathfrak{g})$, preserve (2.1) (i.e. transform solutions into solutions). The left-right symmetry of the RE is to be regarded as a quantum counterpart of the left-right $\mathfrak{g}^L \otimes \mathfrak{g}^R$-symmetry of the target group manifold, the horizontal component of the (Kac–Moody) $\widehat{\mathfrak{g}}_\kappa^L \otimes \widehat{\mathfrak{g}}_\kappa^R$-symmetry of the WZW model in the bulk.

[2] Displaying the indices explicitly, the RE reads (here, $M_{ij} \equiv M_{ij}^\Lambda$)

$$(\mathrm{RE})_{ij,kl}: \qquad R_{kc,ia}\ M_{ab}^{\Lambda_1}\ R_{bj,cd}\ M_{dl}^{\Lambda_2} = M_{kc}^{\Lambda_2}\ R_{cd,ia}\ M_{ab}^{\Lambda_1}\ R_{bj,dl}.$$

The indices $\{i, j, a, b\}$ and $\{k, l, c, d\}$ correspond to the first (1) and the second (2) vector space in (2.1), respectively.

[3] π_Λ is the irreducible representation of $\mathcal{U}_q(\mathfrak{g})$ of the highest weight Λ, a dominant integral affine weight of $\mathfrak{g}$. We denote the set of all such weights (the fundamental affine alcove) by $P_+^\kappa(\mathfrak{g})$. In particular π_{Λ_F} stands for the fundamental (defining) representation.

JHEP04(2006)009

Given the REA defined by the above commutation relations, together with the additional quantum-determinant constraint ($M \equiv M^{\Lambda_F}$):

$$\det{}_q M \overset{!}{\propto} 1, \tag{2.3}$$

to be interpreted as fixing the “size” of the quantum group manifold, we may subsequently consider its irreducible representations separated[4] by the Casimir operators:

$$\mathfrak{c}_k^\Lambda = (\mathrm{tr}_q \otimes \mathrm{id})(M^\Lambda)^k. \tag{2.4}$$

Upon descending to any specific such representation, we break the original left-right symmetry down to the diagonal part:

$$\mathcal{U}_q(\mathfrak{g}^L \times \mathfrak{g}^R)_\mathcal{R} \ni u_L \otimes u_R \searrow u \otimes u \in (\mathcal{U}_q(\mathfrak{g}^L \times \mathfrak{g}^R)_\mathcal{R})^V \cong \mathcal{U}_q(\mathfrak{g}) \tag{2.5}$$

which fits in well with the picture of reduction of the bulk symmetry at an untwisted maximally symmetric boundary[5]. This elementary observation already hints at the viable identification of irreducible representations of the REA with (untwisted) maximally symmetric boundary conditions of the relevant WZW model, that is with (untwisted) maximally symmetric D-branes.

In order to give some flesh to the last statement, we need an explicit realisation of the defining relations (2.1) and (2.3). Luckily, one particular such realisation has long been known [5, 11] (M_0^Λ is an arbitrary c-number solution to the RE):

$$M^\Lambda = L^+ M_0^\Lambda S L^- \equiv (\pi_\Lambda \otimes \mathrm{id})(\mathcal{R}_{21}) M_0^\Lambda (\pi_\Lambda \otimes \mathrm{id})(\mathcal{R}_{12}) \tag{2.6}$$

and is determined by the Faddeev–Reshetikhin–Takhtajan (FRT) realisation [12]:

$$L_{ij}^+ = [(\pi_\Lambda)_{ij} \otimes \mathrm{id}](\mathcal{R}_{21}) \qquad , \qquad L_{ji}^- = [(\pi_\Lambda)_{ij} \otimes \mathrm{id}](\mathcal{R}_{12}^{-1}), \qquad 1 \le i \le j \le \dim V_\Lambda \tag{2.7}$$

of the Drinfel'd–Jimbo quantum group $\mathcal{U}_q(\mathfrak{g})$ in terms of the so-called $L^\pm$-operators satisfying the standard R-matrix commutation relations. *Choosing* this realisation for the REA renders at our disposal the well-developed representation theory of $\mathcal{U}_q(\mathfrak{g})$ whose peculiarities at the CFT-dictated root-of-unity value of the deformation parameter q have provided ample evidence for an intimate relationship between the REA thus reconstructed and quantum D-branes, as summarised below:

- irreducible representations π_Λ of $\mathcal{U}_q(\mathfrak{g})$ of a non-vanishing quantum dimension ($\Lambda \in P_+^\kappa(\mathfrak{g})$) are in one-to-one correspondence with the inequivalent untwisted maximally symmetric boundary states $|\Lambda\rangle\rangle_C$ of the WZW model (a twisted variant of the correspondence has been worked out in [10]), the truncated tensor product of these representations reproduces the fusion rules of the latter - cf [1];

- the representation theory of the REA induced from that of the quantum group accounts well for the discrete symmetries of the group manifold generated by the simple currents of the CFT - cf [9];

[4]Cf [9] for a discussion of (geometrically well-understood) degeneracies.

[5]For an extension to the twisted case, see: [10].

- harmonic analysis on the quantum geometries associated with the irreducible representations of the REA agrees with the decoupling limit [3, 13] of the subalgebra of the boundary OPE algebra composed of horizontal multiplets descended from primary boundary fields that do not change the boundary condition - cf [1];

- localisation of D-branes within the quantum group manifold from fixing Casimir eigenvalues is in keeping with the semiclassical results - cf [1, 10];

- exact values of D-brane tensions follow from a general matrix-action *Ansatz* - cf [1];

- a well-defined semiclassical limit coincides with the perturbative fuzzy structure of [3] - cf [1];

- fractionation of D-branes at fixed-points of simple-current orbifold action admits a straightforward algebraic description - cf [9].

In this paper, we attempt to give an independent justification of the choice of realisation of the REA that underlies the hitherto successful quantum reconstruction programme, whereby we also discover an algebraic description of the D-brane condensation phenomena responsible for creation of arbitrary D-branes of the model (of the kind described) from gauge-field-perturbed stacks of elementary D0-branes. Last, we rederive harmonic analysis on any given D-brane from the analysis of an algebraic fusion procedure and - most importantly - extract from the FRT-realised REA the microscopic D-brane geometry data, as encoded in the graviton coupling to its worldvolume.

3. Bound-state fusion

In this section, we establish a non-trivial link between effective D-brane gauge dynamics in boundary WZW models and the REA's defined by (2.1). We want to introduce an algebraic cousin of the D-brane condensation effect [14], discussed at great length in [3, 15] with reference to the seminal papers by Affleck and Ludwig [16]. In the case at hand, the very form of fusion leads us to conclude that an arbitrary D-brane, as described by its function algebra, can be built out of a number of trivial representations of the RE, describing D0-branes. The algebraic fusion algorithm has been devised in direct reference to the techniques of the principal chiral model presented in [17], in which there is additional structure (dependence on a dynamical parameter) justifying its interpretation. In the present setup, lacking this extra structure, some elementary tests of its validity are performed explicitly below, as well as in the Appendix. In particular, we verify - rather importantly - that it has the expected semiclassical limit.

We begin by remarking that the operator-valued matrix[6] $M^{\Lambda(\Lambda_B)}$ given by either side of (2.1),

$$M_1^{\Lambda(\Lambda_B)} = M_2^{\Lambda_B} R_{21}^{\Lambda,\Lambda_B} M_1^{\Lambda} R_{12}^{\Lambda,\Lambda_B}, \tag{3.1}$$

[6]Here, we add an extra index (Λ_B) which indicates that $M^{\Lambda(\Lambda_B)} \in \mathrm{End}(V_\Lambda) \otimes \mathrm{End}(V_{\Lambda_B}) \otimes \mathrm{REA}_q(\mathfrak{g})$. Earlier analyses focused mainly on the special case $(\Lambda, \Lambda_B) = (\Lambda_F, 0)$ in which $M^{\Lambda_F(0)} \equiv M^{\Lambda_F}$ represents the coordinate module of the quantised group manifold. The meaning of M^Λ for a general Λ shall be expounded in the next section.

also satisfies an appropriate RE:

$$R_{21}^{\Lambda_1,\Lambda_2} M_1^{\Lambda_1(\Lambda_B)} R_{12}^{\Lambda_1,\Lambda_2} M_2^{\Lambda_2(\Lambda_B)} = M_2^{\Lambda_2(\Lambda_B)} R_{21}^{\Lambda_1,\Lambda_2} M_1^{\Lambda_1(\Lambda_B)} R_{12}^{\Lambda_1,\Lambda_2} \tag{3.2}$$

for Λ_B arbitrary. The latter follows straightforwardly from the RE's and the Quantum Yang–Baxter Equation satisfied by the M-matrices fused and the $\mathcal{R}$-matrix, respectively. In other words, the Bound-State Fusion (BSF) thus defined, (3.1), provides a systematic method of generating new solutions to the RE from the known ones.

The physical significance of (3.1) relies on the observation that it singles out a set of REA representations of special relevance to the study of WZW D-branes. Take any c-number matrix $(M_0)^\Lambda$ respecting the RE (considered, e.g., in [5]) so that $(M_0)_2^{\Lambda_B}(M_0)_1^\Lambda$ also satisfies the RE. According to the logic outlined in Sec.2, the latter is - for $(M_0)^{\Lambda_B} = \mathbb{I}$ (the unit matrix of dimension $\dim V_{\Lambda_B}$) - to be associated with $\dim V_{\Lambda_B}$ D0-branes located at positions defined by $(M_0)^\Lambda$ as *per* Casimir eigenvalues (2.4). Then, the right-hand side of (3.1) belongs to $(\pi_\Lambda \otimes \pi_{\Lambda_B})(\mathcal{U}_q(\mathfrak{g}) \otimes \mathcal{U}_q(\mathfrak{g}))$. We interpret the process of passing from the reducible representation just described, $(M_0)_2^{\Lambda_B}(M_0)_1^\Lambda$, to the irreducible one given by (3.1) as condensation and depict it as

$$(M_0)_2^{\Lambda_B}(M_0)_1^\Lambda \;\longrightarrow\; M^{\Lambda(\Lambda_B)} \equiv R_{21}^{\Lambda,\Lambda_B}(M_0)_1^\Lambda R_{12}^{\Lambda,\Lambda_B}. \tag{3.3}$$

This, however, is none other but the FRT realisation (2.6) of the irreducible representation of $\mathrm{REA}_q(\mathfrak{g}) \hookrightarrow \mathcal{U}_q(\mathfrak{g})$ of highest weight Λ_B, *chosen* in [1] for the simple reason: it induces a representation theory of $\mathrm{REA}_q(\mathfrak{g})$ whose elements, irreducible highest-weight representations of $\mathcal{U}_q(\mathfrak{g})$ of a non-vanishing quantum dimension, are in a straightforward one-to-one correspondence with all the candidate algebraic D-branes associated, in [13, 18], with untwisted maximally symmetric WZW boundary conditions on the compact (simple and simply connected) Lie group G. Thus, we can postulate the following

Principle. *Untwisted maximally symmetric quantum WZW D-branes on a simple and simply connected compact Lie group G are classified by those irreducible representations of $REA_q(\mathfrak{g})$ which can be generated through the Bound-State Fusion (3.1) from an elementary c-number D0-brane solution.*

The BCFT-preferred FRT realisation is now an immediate consequence of the Principle whose physical rationale shall be presented below.

We may next apply the fusion algorithm to the physical solutions generated from the D0-brane one. Thus, given $M^{\Lambda_B(\lambda)}$ and $M^{\Lambda(\lambda)}$ the fusion (3.1) leads to:

$$M^{\Lambda(\Lambda_B\times\lambda)} = M_2^{\Lambda_B(\lambda)} R_{21}^{\Lambda,\Lambda_B} M_1^{\Lambda(\lambda)} R_{12}^{\Lambda,\Lambda_B}. \tag{3.4}$$

Here, the left-hand side belongs to $[\pi_\Lambda \otimes (\pi_{\Lambda_B} \otimes \pi_\lambda)](\mathcal{U}_q(\mathfrak{g}) \otimes \mathcal{U}_q(\mathfrak{g}) \otimes \mathcal{U}_q(\mathfrak{g}))$ and hence - as a tensor operator[7] - it can be decomposed as

$$M^{\Lambda(\Lambda_B\times\lambda)} = \oplus_{\mu\in P_+^\kappa(\mathfrak{g})} \mathcal{N}_{\Lambda_B\,\lambda}^{\;\;\;\;\mu} M^{\Lambda(\mu)}, \tag{3.5}$$

[7]Cp [1] and the Appendix.

where $\mathcal{N}_{\Lambda_B}{}^{\mu}_{\lambda}$ are the standard fusion rules of the WZW model with the current symmetry $\widehat{\mathfrak{g}}_\kappa$ and the usual restriction to irreducible representations of $\mathcal{U}_q(\mathfrak{g})$ of a non-vanishing quantum dimension has been imposed.

The last result as well as the reasoning that has led us to formulate the Principle are strongly reminiscent of the BCFT picture in which gauge-field perturbations induce transitions through condensation between an original stack of D-branes and a final (metastable) state. Let us dwell on this a little longer.

The fusion operation:

$$M^{\Lambda(\lambda)} \xrightarrow{\Lambda_B - BSF} M^{\Lambda(\Lambda_B \times \lambda)} \tag{3.6}$$

defined above mimics the BCFT transition [15]:

$$(\lambda; \dim\Lambda_B) \xrightarrow{A^{\Lambda_B}} \oplus_{\mu \in P_+^\kappa(\mathfrak{g})} \mathcal{N}_{\Lambda_B}{}^{\mu}_{\lambda} (\mu; 1), \tag{3.7}$$

of a stacked $\dim V_{\Lambda_B}$-tuple of D-branes of weight label λ, effected by the marginal perturbation: $\int_{\partial\Sigma} dt A_a^{\Lambda_B} J^a(t)$ ($\partial\Sigma$ is the boundary of the open-string worldsheet) of the boundary WZW model coupling the *constant* gauge field $A_a^{\Lambda_B} = \pi_{\Lambda_B}(T_a) \otimes \mathbb{I}_{d_\lambda}$[8] to the boundary symmetry current J. In the relevant fuzzy matrix model [3], the transition is realised by perturbing the background geometry $Y_a^{\lambda, d_{\Lambda_B}} = \mathbb{I}_{d_{\Lambda_B}} \otimes \pi_\lambda(T_a)$ of a stack of (d_{Λ_B}) fuzzy D-branes of weight label λ with the specific gauge fluctuation $A_a^{\Lambda_B}$ as

$$Y_a^{\lambda, d_{\Lambda_B}} \xrightarrow{A^{\Lambda_B}} Y_a^{\lambda, d_{\Lambda_B}} + A_a^{\Lambda_B} = \oplus_{\mu \in P_+(\mathfrak{g})} L_{\Lambda_B}{}^{\mu}_{\lambda} Y_a^{\mu,1}, \tag{3.8}$$

whereby a semiclassical (large-κ) variant of the condensation effect is induced ($L_{\Lambda_B}{}^{\mu}_{\lambda}$ are the Littlewood–Richardson coefficients which replace the fusion rules at large values of the level). Motivated thus, we put forward the following

Claim. *The Bound-State Fusion (3.4) captures — in the algebraic framework of the REA — the gauge-field-driven effect of condensation with boundary-spin absorption (3.7).*

In order to substantiate it, we need to go back to [8] and identify nontrivial gauge-field degrees of freedom on a stack of quantum D-branes. Hence, we associate small (we have a natural expansion parameter $\hbar \equiv \frac{\pi}{\kappa + g^\vee(\mathfrak{g})}$) gauge-field excitations - in the vein of a much more general approach to gauge fields on a noncommutative geometry - with perturbations of the geometric background, $M^{\Lambda(\lambda)}$, $\Lambda = \Lambda_F$ (the coordinate module), exactly as in the semiclassical picture (3.8). Furthermore, we decompose some of the terms in (3.4), $X \in \{M_2^{\Lambda_B(\lambda)}, R_{21}^{\Lambda_F, \Lambda_B}, R_{12}^{\Lambda_F, \Lambda_B}\}$, as $X = \mathbb{I} \otimes \mathbb{I} + x$, where x is of the order of $\mathcal{O}(\hbar)$ and (q-)traceless (up to corrections of higher order in $\hbar$). With this decomposition, in which we assume[9] that $0 \lesssim \|\lambda\|, \|\Lambda_B\| \ll \kappa$, the leading term in (3.4) reads $\sigma_{1,2}(\mathbb{I}_{d_{\Lambda_B}} \otimes M^{\Lambda_F(\lambda)})$ ($\sigma_{1,2}$ interchanges the first and second tensor components) and shall be denoted by $M_0^{\Lambda_F(\Lambda_B \times \lambda)}$. Thus, the right-hand side of (3.4) can be rewriten as

$$M^{\Lambda_F(\Lambda_B \times \lambda)} = M_0^{\Lambda_F(\Lambda_B \times \lambda)} + A^{\Lambda_F(\Lambda_B \times \lambda)}, \tag{3.9}$$

[8] $\pi_{\Lambda_B}(T_a)$ are the generators of the horizontal Lie algebra $\mathfrak{g}$, satisfying the defining relation $[T_a, T_b] = 2\imath f_{abc} T_c$, and realised in the representation π_{Λ_B}; moreover, we have denoted $d_\lambda := \dim V_\lambda$.

[9] This is precisely the domain of validity of the semiclassical approximation.

where $A^{\Lambda_F(\Lambda_B\times\lambda)} \sim \mathcal{O}(\hbar)$ acquires the interpretation of a gauge field[10]. Precise agreement between our description[11] and the BCFT one (3.8) follows from the fact that the RE (3.2), satisfied by $M^{\Lambda_F(\Lambda_B\times\lambda)}$, reproduces — in the semiclassical régime, at $\mathcal{O}(\hbar)$ - exactly the matrix equations of the fuzzy model of the BCFT [1] satisfied by the right-hand side of (3.8), that is the vanishing-curvature equation for the gauge potential A^{Λ_B} (in this picture, the semiclassical transformation rules for A^{Λ_B} become a consequence of those of the covariant coordinate $M^{\Lambda_F(\Lambda_B\times\lambda)}$). Alternatively, one may perform an $\hbar$-expansion of the explicit FRT realisation of $M^{\Lambda_F(\Lambda_B\times\lambda)}$, whereby one readily reobtains (3.8) at the first nontrivial level.

4. Bulk-weight fusion and brane-gravity couplings

Let us begin by recalling that the quantised algebra of functions on untwisted D-branes, $\mathrm{REA}_q(\mathfrak{g})$, is generated by the elements $M^{\Lambda_F}_{ij}$. There is a natural basis of the algebra, regarded here as a vector space, namely the basis of $\mathcal{U}_q(\mathfrak{g})$-intertwiners related directly - in the physical context - to the multiplets of boundary fields on a given D-brane descended from the primary fields of the BCFT by the action of the horizontal subalgebra $\mathfrak{g}$ of the current symmetry algebra $\widehat{\mathfrak{g}}_\kappa$ of the relevant WZW model (in the decoupling limit of [3, 13]). We claim that

$$\mathrm{REA}_q(\mathfrak{g}) = \oplus_{\Lambda\in P^\kappa_+(\mathfrak{g})}\mathrm{span}\langle M^\Lambda_{ij}\rangle_{i,j\in\overline{1,d_\Lambda}}, \tag{4.1}$$

where M^Λ_{ij} is the (i,j)-th operator entry of the matrix M^Λ, is the basis sought. Above, M^Λ denote matrices respecting the RE (2.1) written in the representation $\pi_\Lambda\otimes\pi_\Lambda$. Thus, $M^{\Lambda(\lambda)}$ (see the previous section) are - indeed - quantum-group tensors with transformation properties appropriate for functions on the standard set of D-branes,

$$(1\otimes\pi_\lambda(u_1))M^{\Lambda(\lambda)}(1\otimes\pi_\lambda(Su_2)) = (\pi_\Lambda(Su_1)\otimes 1)M^{\Lambda(\lambda)}(\pi_\Lambda(u_2)\otimes 1). \tag{4.2}$$

The road to (4.1) goes through the definition of the Bulk-Weight Fusion (BWF):

$$M^{\Lambda_1\times\Lambda_2}_{12} = \left(R^{\Lambda_1,\Lambda_2}_{12}\right)^{-1} M^{\Lambda_1}_1 R^{\Lambda_1,\Lambda_2}_{12} M^{\Lambda_2}_2. \tag{4.3}$$

The fusion is a solution-generating operation for the "bulk" (matrix) tensor component of $M^{\Lambda(\lambda)}$, compatible with the defining relation (2.1) of a $\mathcal{U}_q(\mathfrak{g}^L\times\mathfrak{g}^R)_\mathcal{R}$-module [11]. It is a natural counterpart of the classical tensoring procedure[12] in the category of solutions to the RE - one can easily show that (4.3) yields a $(\pi_{\Lambda_1}\otimes\pi_{\Lambda_2})(\mathcal{U}_q(\mathfrak{g}^L\times\mathfrak{g}^R)_\mathcal{R}\otimes\mathcal{U}_q(\mathfrak{g}^L\times\mathfrak{g}^R)_\mathcal{R})$-module and respects the corresponding RE[13]; the right-hand side of (4.3) can be projected

[10] Using the BWF of Sec.4, the gauge field is readily shown to be a "function" of the background geometry $M^{\Lambda_F(\lambda)}$.

[11] It is worth remarking that the other natural (given the FRT realisation) candidate for an algebraic description of the condensation phenomena, namely the standard coproduct in the second tensor component of $\mathcal{M} = \mathcal{R}_{21}\mathcal{R}_{12}$ in (2.6), yields essentially the same result.

[12] It is, in particular, equivalent to the standard coproduct in the first tensor component of the universal M-matrix $\mathcal{M} = \mathcal{R}_{21}\mathcal{R}_{12}$ in the FRT realisation.

[13] The RE in question is (2.1) with both π_{Λ_1} and π_{Λ_2} replaced by $\pi_{\Lambda_1}\otimes\pi_{\Lambda_2}$.

JHEP04(2006)009

onto irreducible components, M^Λ, with $\pi_\Lambda \subset \pi_{\Lambda_1} \otimes \pi_{\Lambda_2}$. Thus, starting from M^{Λ_F} (π_{Λ_F} is the defining representation of $\mathcal{U}_q(\mathfrak{g})$) we can generate a basis of matrices M^Λ for any $\Lambda \in P_+^\kappa(\mathfrak{g})$. This leads directly to (4.1).

The above iterative algorithm for obtaining tensor-product solutions from some given elementary ones, M^{Λ_1} and M^{Λ_2}, is our second example of RE fusion and was discussed at great length, in the above form, in [11]. As we already know, it is not the only way of composing elements of $\mathrm{REA}_q(\mathfrak{g})$. We shall therefore distinguish it by giving it a name suggested (once more) by the literature on the $(1+1)$-dimensional models, that is the Bulk-Weight Fusion.

Below, we give an interpretation to $\pi_\lambda(M^\Lambda_{ij})$. Recall that $\pi_\lambda(M^\Lambda_{ij}) \in \mathrm{End}(V_\lambda)$ for the D-brane labeled by the weight λ. Accordingly, we may calculate the (q-)trace of M^Λ over the module V_λ. It is straightforward to demonstrate [1] that the trace is proportional to the unit matrix, that is

$$\mathrm{tr}_q^{(\lambda)}(M^\Lambda_{ij}) = \mathrm{tr}_{V_\lambda}(M^\Lambda_{ij} \cdot q^{2H_\rho}) = f(\Lambda,\lambda)\ \delta_{ij}, \tag{4.4}$$

where ρ is the Weyl vector of $\mathfrak{g}$. As shown in [1], $\pi_\lambda(M^\Lambda_{ij})$ in the FRT realisation (3.3) encode a number of properties of the weight-λ D-brane in an algebraic manner (cp Sec.1). Since we have not normalised M so far, we can specify the function $f(\Lambda,\lambda)$ up to a λ-dependent factor only. Let us calculate $f(\Lambda,\lambda)$. We use [19]

$$\mathcal{R}_{12} = q^{H_i F_{ij} \otimes H_j} \left(I \otimes I + \sum_{U^\pm} U^+ \otimes U^- \right), \tag{4.5}$$

while for $\mathcal{R}_{21}$ we transpose $U^+ \leftrightarrow U^-$ in the expression above. Here, F is the (symmetric) quadratic matrix of $\mathfrak{g}$, and U^+, U^- stand for terms in the Borel subalgebras of rising and lowering operators, respectively. As the left-hand side of (4.4) does not depend on the vector from the module V_Λ that it acts upon, we can evaluate it on the highest-weight vector (annihilated by U^+), $|\Lambda\rangle$. Then, only the generators of the Cartan subalgebra in (4.5) contribute,

$$q^{2H_i F_{ij} \otimes H_j}|_{|\Lambda\rangle \otimes \cdot} = q^{2\Lambda_i F_{ij} \otimes H_j} = I \otimes q^{2H_\Lambda}, \tag{4.6}$$

so that (4.4) becomes[14]

$$\mathrm{tr}_q^{(\lambda)}(M^\Lambda_{ij}) = \delta_{ij}\ \mathrm{tr}_{V_\lambda}(q^{2(H_\Lambda + H_\rho)}) = \delta_{ij}\ \chi_\lambda \left(\frac{2\pi i(\Lambda+\rho)}{\kappa + g^\vee(\mathfrak{g})} \right) = \delta_{ij}\ \frac{S_{\Lambda^+\lambda}}{S_{\Lambda^+0}}, \tag{4.7}$$

where χ_λ and $S_{\Lambda^+\lambda}$ are the standard character over the $\mathfrak{g}$-module of highest weight λ and the modular matrix of the WZW model associated to $\widehat{\mathfrak{g}}_\kappa$, respectively, whereas Λ^+ is the unique charge conjugate of the weight Λ. In particular, for $\mathfrak{g} = \mathfrak{su}_2$, we obtain

$$f_{\mathfrak{su}_2}(\Lambda,\lambda) = \mathrm{tr}_{V_\lambda}(q^{(\Lambda+1)H}) = \frac{\sin \frac{\pi(\Lambda+1)(\lambda+1)}{k+2}}{\sin \frac{\pi(\Lambda+1)}{k+2}}, \tag{4.8}$$

which agrees with $S^{\mathfrak{su}_2}_{\Lambda\lambda} = \sqrt{\frac{2}{k+2}} \sin \frac{\pi(\Lambda+1)(\lambda+1)}{k+2}$ and $\Lambda^+ \equiv \Lambda$ for all Λ.

[14]The explicit formula relating entries of the modular S-matrix to Lie-algebra characters can be found, e.g., in [20].

In order to understand the physics behind the last result, recall that - on the BCFT side - untwisted maximally symmetric D-branes of the WZW model are represented by Cardy states [21]:

$$|\lambda\rangle\rangle_C = \sum_{\Lambda \in P_+^\kappa(\mathfrak{g})} \frac{S_{\Lambda\lambda}}{\sqrt{S_{\Lambda 0}}} \, |\Lambda\rangle\rangle_I. \tag{4.9}$$

Above, $|\Lambda\rangle\rangle_I$ are the Ishibashi (character) states [22]. The data encoded in (4.9) turn out to be sufficient to determine, to the leading order in $\hbar$, the coupling of **graviton** modes:

$$|a, b, \gamma_{ij}\rangle = J_{-1}^{(a} \tilde{J}_{-1}^{b)} |\gamma_i\rangle \otimes |\gamma_j^+\rangle, \qquad |\gamma_i\rangle \otimes |\gamma_j^+\rangle \in \widehat{V}_\gamma \otimes \widehat{V}_{\gamma^+} \tag{4.10}$$

to the D-brane defined by (4.9) (here, J_{-1} and $\tilde{J}_{-1}$ are the (-1)-th Laurent modes of the two chiral components of the bulk $\widehat{\mathfrak{g}}_\kappa^L \otimes \widehat{\mathfrak{g}}_\kappa^R$-symmetry current, acting on the $\widehat{\mathfrak{g}}_\kappa$ modules $\widehat{V}_{\gamma,\gamma^+}$ of highest weights $\gamma, \gamma^+ \in P_+^\kappa(\mathfrak{g})$, respectively). Indeed, one readily verifies that ($\mathcal{N}$ is an irrelevant normalisation constant)

$$\langle a, b, \gamma_{ij} | \lambda\rangle\rangle_C = \mathcal{N} \delta^{ab} \delta_{ij} \, \frac{S_{\gamma\lambda}}{\sqrt{S_{\gamma 0}}}. \tag{4.11}$$

In the present context, we are dealing with a matrix model whose elementary degrees of freedom are D0-branes (the D0-brane enters the quantum-algebraic construction as the trivial representation of $\mathrm{REA}_q(\mathfrak{g})$, on which $M^{\Lambda(0)} = \mathbb{I}_{d_\Lambda}$), hence it seems only natural to consider couplings normalised relative the reference D0-brane,

$$\frac{\langle a, a, \gamma_{ij} | \lambda\rangle\rangle_C}{\langle b, b, \gamma_{kk} | 0\rangle\rangle_C} = \delta_{ij} \, \frac{S_{\gamma\lambda}}{S_{\gamma 0}}. \tag{4.12}$$

From direct comparison between (4.7) and (4.12), we then draw the following

Conclusion. *The $\mathcal{U}_q(\mathfrak{g})$-tensor operators $M^{\Lambda(\lambda)}$, $\Lambda \in P_+^\kappa(\mathfrak{g})$, obtained from elementary solutions to the Reflection Equation through the Boundary-Weight Fusion* (4.3) *and composing a physically distinguished basis of the algebra of functions on the untwisted maximally symmetric WZW D-brane labeled by the weight $\lambda \in P_+^\kappa(\mathfrak{g})$, encode the complete information on bulk graviton $|a, b, \Lambda_{ij}^+\rangle$ couplings to the Cardy boundary state $|\lambda\rangle\rangle_C$ representing the D-brane, relative an elementary D0-brane, as expressed by the identity:*

$$\frac{\langle a, a, \Lambda_{ij}^+ | \lambda\rangle\rangle_C}{\langle b, b, \Lambda_{kk}^+ | 0\rangle\rangle_C} = \mathrm{tr}_q^{(\lambda)}(M_{ij}^\Lambda). \tag{4.13}$$

We emphasise that it is not just the numerical values of the couplings but also their structure, diagonal in the bulk representation indices, that can be read off from (4.7). The D0-brane data (e.g. the D0-brane tension), on the other hand, have to be supplemented independently of the algebra.[15]

[15] Actually, on the level of bulk-boundary couplings, the only piece of data that cannot be retrieved from the algebra is S_{00}. Indeed, we have [20]

$$S_{\Lambda^+\lambda} = S_{0\Lambda} \cdot \chi_\lambda\left(\frac{2\pi i(\Lambda+\rho)}{\kappa + g^\vee(\mathfrak{g})}\right) = S_{00} \cdot \chi_\Lambda\left(\frac{2\pi i \rho}{\kappa + g^\vee(\mathfrak{g})}\right) \cdot \chi_\lambda\left(\frac{2\pi i(\Lambda+\rho)}{\kappa + g^\vee(\mathfrak{g})}\right) = S_{00} \cdot \mathrm{tr}_q^{(\Lambda)}\left(\mathrm{tr}_q^{(\lambda)} M^\Lambda\right), \tag{4.14}$$

where we have first used the symmetries of the modular S-matrix: $S_{\lambda^+\mu} = S_{\lambda\mu^+}$ and $S_{\lambda\mu} = S_{\mu\lambda}$, and later reiterated the first equality.

5. Conclusions

In the present paper, we have discussed several application of the algebraic RE fusion to the description of physics of untwisted maximally symmetric WZW D-branes. The Bound-State Fusion has been shown to lead to the appropriate choice of realisations of the REA and to give a nice picture of higher-dimensional quantum D-branes as condensates of the elementary quantum D0-branes, reproducing - in the semiclassical approximation - precisely the fuzzy condensation scenario derived from stringy perturbation theory in [3]. It also seems to offer some insight into the structure of gauge fluctuations of the non-commutative geometry defined by the quantised function algebra $\mathrm{REA}_q(\mathfrak{g})$: being a multiplicative perturbation of this background geometry, the gauge fluctuations are strongly reminiscent of the nonperturbative Wilson-loop operators of Bachas and Gaberdiel [23], which - in turn - fits in well with earlier findings on the rôle of (open) Wilson lines in gauge field theories on noncommutative geometries [24]. The Bulk-Weight Fusion, on the other hand, has been demonstrated to encode a fairly complete information on the gravitational D-brane couplings. Both are amazingly simple and follow straightforwardly from the structure of the RE.

In spite of the progress, signified by our results, in formulating a compact description of quantum WZW geometry and elucidating the quantum-group structure of the associated BCFT, a lot more still needs to be understood in this context. We hope to return to these riddles soon.

Acknowledgments

The authors would like to thank the organisers of the 2004 ESI Workshop on "String theory on non-compact and time-dependent backgrounds" where part of this work was done. Our gratitude also goes to the Theoretical Physics Group at King's College London and the String Theory Group at Queen Mary London, and in particular to Thomas Quella, Sanjaye Ramgoolam, Andreas Recknagel, Sylvain Ribault and Steven Thomas, for their sustained interest in the project and stimulating discussions. R.R.S. is thankful to Krzysztof Gawędzki for many illuminating discussions, and - in particular - for explaining to him the Poisson structure behind the algebras of interest and for drawing his attention to the papers expounding the rôle it plays in the canonical description of the (boundary) WZW model. Both authors gratefully acknowledge useful remarks from the referee.

A. Quantum-group covariance

Below, we consider $\mathcal{U}_q(\mathfrak{g})$-covariance properties of the various generalised reflection matrices introduced in the main text. In particular, we give a simple proof of (4.2) and (4.4), essentially repeating the original one from [1]. First, we show, for $\mathcal{M}_{12} := \mathcal{R}_{21}\mathcal{R}_{12}$,

$$\Delta(u)\mathcal{M}_{12} = \Delta(u)\mathcal{R}_{21}\mathcal{R}_{12} = \mathcal{R}_{21}\Delta^{cop}(u)\mathcal{R}_{12} = \mathcal{R}_{21}\mathcal{R}_{12}\Delta(u) = \mathcal{M}_{12}\Delta(u), \tag{A.1}$$

where we have invoked the twisting property of $\mathcal{R}$ [25, 26]:

$$\Delta^{cop}(u) = \mathcal{R}\Delta(u)\mathcal{R}^{-1}, \qquad \Delta^{cop}(u) := u_2 \otimes u_1. \tag{A.2}$$

Using Hopf-algebra identities for the coproduct and the antipode of $\mathcal{U}_q(\mathfrak{g})$ (i.e. taking (A.1) with both sides of the identity extended by $Su_0 \otimes I$ from the left and by $I \otimes Su_3$ from the right, and - upon contracting and then multiplying the tensor factors in the pairs of spaces $(0,1)$ and $(2,3)$ - representing both sides on $V_\Lambda \otimes V_\lambda$), we turn (A.1) into (4.2), or

$$\pi_\lambda(u_1) M^{\Lambda(\lambda)} \pi_\lambda(Su_2) = \pi_\Lambda(Su_1) M^{\Lambda(\lambda)} \pi_\Lambda(u_2), \tag{A.3}$$

for any $u \in \mathcal{U}_q(\mathfrak{g})$.

In order to prove (4.4), we recall the definition of the quantum trace: $\mathrm{tr}_q(x) := \mathrm{tr}(x\mathrm{v})$, where $\mathrm{v} := (S \otimes \mathrm{id})(\mathcal{R}_{21})$ is the distinguished invertible ($\mathrm{v}^{-1} \equiv S\mathrm{v}$) element of $\mathcal{U}_q(\mathfrak{g})$ satisfying $S^2 u = \mathrm{v}u\mathrm{v}^{-1}$ for any $u \in \mathcal{U}_q(\mathfrak{g})$ [25]. This, together with (4.2), immediately implies

$$\pi_\Lambda(Su_1)\mathrm{tr}_q^{(\lambda)}(M^\Lambda)\pi_\Lambda(u_2) = \mathrm{tr}_q^{(\lambda)}(u_1 M^\Lambda Su_2) = \mathrm{tr}_q^{(\lambda)}(M^\Lambda Su_2 \mathrm{v} u_1) = \varepsilon(u)\mathrm{tr}_q^{(\lambda)}(M^\Lambda), \tag{A.4}$$

or, equivalently,

$$[\pi_\Lambda(u), \mathrm{tr}_q^{(\lambda)}(M^\Lambda)] = 0. \tag{A.5}$$

Last, we may verify the tensorial character of (3.1), on which our physical interpretation of the BSF has been based. Our proof is in fact a slight variation of the trick used above. We begin by defining the operator $\mathcal{M}_{123} = \mathcal{R}_{32}\mathcal{R}_{23}\mathcal{R}_{21}\mathcal{R}_{31}\mathcal{R}_{13}\mathcal{R}_{12}$ such that $M_{1\ 2\otimes 3}^{\Lambda,\Lambda_B\times\lambda} \equiv (\pi_\Lambda \otimes \pi_{\Lambda_B} \otimes \pi_\lambda)(\mathcal{M}_{123})$. Using (A.2) again, we then obtain

$$[(\Delta \otimes \mathrm{id}) \otimes \Delta](u)\mathcal{M}_{123} = \mathcal{M}_{123}[(\Delta \otimes \mathrm{id}) \otimes \Delta](u) \tag{A.6}$$

and hence

$$(Su_1 \otimes I \otimes I)\mathcal{M}_{123}(u_2 \otimes I \otimes I) = (I \otimes u_1 \otimes u_2)\mathcal{M}_{123}(I \otimes Su_4 \otimes Su_3). \tag{A.7}$$

The latter formula ultimately turns into an appropriate analogon of (A.3),

$$\pi_\Lambda(Su_1)M^{\Lambda,\Lambda_B\times\lambda}\pi_\Lambda(u_2) = \pi_{\Lambda_B\otimes\lambda}(u_1)M^{\Lambda,\Lambda_B\times\lambda}\pi_{\Lambda_B\otimes\lambda}(Su_2), \tag{A.8}$$

once we invoke one of the fundamental properties of a Hopf algebra [25], $\Delta \circ S = (S \otimes S) \circ \Delta^{cop}$, and use the standard definition of a tensor-product representation of a coalgebra, $\pi_{\Lambda_1\otimes\Lambda_2} := (\pi_{\Lambda_1} \otimes \pi_{\Lambda_2}) \circ \Delta$.

References

[1] J. Pawełczyk and H. Steinacker, *Matrix description of D-branes on 3-spheres*, *JHEP* **12** (2001) 018 [hep-th/0107265]; *A quantum algebraic description of D-branes on group manifolds*, *Nucl. Phys.* **B 638** (2002) 433 [hep-th/0203110].

[2] V. Schomerus, *Lectures on branes in curved backgrounds*, *Class. and Quant. Grav.* **19** (2002) 5781 [hep-th/0209241].

[3] A.Y. Alekseev, A. Recknagel and V. Schomerus, *Non-commutative world-volume geometries: branes on* SU(2) *and fuzzy spheres*, *JHEP* **09** (1999) 023 [hep-th/9908040]; *Brane dynamics in background fluxes and non-commutative geometry*, *JHEP* **05** (2000) 010 [hep-th/0003187].

[4] E. Sklyanin, *Boundary conditions for integrable quantum systems*, *J.Phys.* **A21** (1988) 2375-2389;
L. Mezincescu and R. I. Nepomechie, *Integrable open spin chains with nonsymmetric R matrices*, *J.Phys.* **A21** (1991) L17-L24; *Integrability of open spin chains with quantum algebra symmetry*, *Int. J. Mod. Phys.* **A 6** (1991) 5231 [hep-th/9206047].

[5] P.P. Kulish, R. Sasaki and C. Schwiebert, *Constant solutions of reflection equations and quantum groups*, *J. Math. Phys.* **34** (1993) 286 [hep-th/9205039];
P.P. Kulish and E.K. Sklyanin, *Algebraic structures related to the reflection equations*, *J. Phys.* **A 25** (1992) 5963 [hep-th/9209054];
P.P. Kulish and R. Sasaki, *Covariance properties of reflection equation algebras*, *Prog. Theor. Phys.* **89** (1993) 741 [hep-th/9212007].

[6] M.A. Semenov-Tian-Shansky, *Poisson Lie groups, quantum duality principle and the quantum double*, *Theor. Math. Phys.* **93** (1992) 1292 [hep-th/9304042].

[7] K. Gawędzki, I. Todorov and P. Tran-Ngoc-Bich, *Canonical quantization of the boundary Wess-Zumino-Witten model*, *Commun. Math. Phys.* **248** (2004) 217 [hep-th/0101170].

[8] J. Pawełczyk and H. Steinacker, *Algebraic brane dynamics on* SU(2)*: excitation spectra*, *JHEP* **12** (2003) 010 [hep-th/0305226].

[9] J. Pawełczyk and R.R. Suszek, *Quantum matrix models for simple current orbifolds*, *Nucl. Phys.* **B 710** (2005) 599 [hep-th/0310289].

[10] J. Pawełczyk, R.R. Suszek and H. Steinacker, *Twisted WZW branes from twisted REA's*, *Nucl. Phys.* **B 724** (2005) 487 [hep-th/0412146].

[11] J. Donin, P. P. Kulish and A. I. Mudrov, *On universal solution to reflection equation*, *Lett. Math. Phys.* **63** (2003) 179 [math.QA/0210242].

[12] L. D. Faddeev, N. Yu. Reshetikhin and L. A. Takhtajan, *Quantization of Lie groups and Lie algebras*, *Lengingrad Math.J.* **1** (1990) 193-225, *Algebra Anal.* **1** (1989) 178-206.

[13] G. Felder, J. Fröhlich, J. Fuchs and C. Schweigert, *The geometry of WZW branes*, *J. Geom. Phys.* **34** (2000) 162 [hep-th/9909030].

[14] R.C. Myers, *Dielectric-branes*, *JHEP* **12** (1999) 022 [hep-th/9910053].

[15] S. Fredenhagen and V. Schomerus, *Branes on group manifolds, gluon condensates and twisted K-theory*, *JHEP* **04** (2001) 007 [hep-th/0012164].

[16] I. Affleck and A. W. W. Ludwig, *The Kondo effect, conformal field theory and fusion rules*, *Nucl. Phys.* **B 352** (1991) 849; *Critical theory of overscreened Kondo fixed points*, *Nucl. Phys.* **B 60** (1991) 641; *Universal noninteger 'ground state degeneracy' in critical quantum systems*, *Phys. Rev. Lett.* **67** (1991) 161.

[17] G.W. Delius, N.J. MacKay and B.J. Short, *Boundary remnant of Yangian symmetry and the structure of rational reflection matrices*, *Phys. Lett.* **B 522** (2001) 335 [hep-th/0109115].

[18] A.Y. Alekseev and V. Schomerus, *D-branes in the WZW model*, *Phys. Rev.* **D 60** (1999) 061901 [hep-th/9812193].

[19] A. Klimyk and K. Schmüdgen, *Quantum groups and their representations*, Springer-Verlag, 1998.

[20] P. Di Francesco, P. Mathieu and D. Sénéchal, *Conformal field theory*, Springer-Verlag, 1997.

JHEP04(2006)009

[21] J. L. Cardy, *Boundary conditions, fusion rules and the Verlinde formula*, *Nucl. Phys.* **B 324** (1989) 581.

[22] N. Ishibashi, *The boundary and crosscap states in conformal field theories*, *Mod. Phys. Lett.* **A 4** (1989) 251.

[23] C. Bachas and M. Gaberdiel, *Loop operators and the Kondo problem*, *JHEP* **11** (2004) 065 [hep-th/0411067].

[24] N. Ishibashi, S. Iso, H. Kawai and Y. Kitazawa, *Wilson loops in noncommutative Yang-Mills*, *Nucl. Phys.* **B 573** (2000) 573 [hep-th/9910004];
D.J. Gross, A. Hashimoto and N. Itzhaki, *Observables of non-commutative gauge theories*, *Adv. Theor. Math. Phys.* **4** (2000) 893 [hep-th/0008075].

[25] S. Majid, *Foundations of quantum group theory*, Cambridge University Press, 1995.

[26] V. Chari and A. Pressley, *A guide to quantum groups*, Cambridge University Press, 1994.